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Weight conservation condition and structure of the braid group representation

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Abstract. A weight conservation condition for the S -matrix of the braid group representation, from which non-vanishing elements of the S -matrix can be determined, is introduced. A method of weight lattice analysis of representations of Lie algebra is suggested and used to give structures of braid group representations in various cases. This makes it feasible for braid group representations to be obtained by solving the spectral parameter independent Yang–Baxter equation directly, which leads to several new braid group representations.

1. Introduction

It is known that the Yang–Baxter equation (YBE) plays a crucial role in the study of exactly solvable models in two-dimensional statistical mechanics, quantum integrable systems and conformal field theory [1–6]. Recently there has been a great deal of interest in the study of braid group representations (BGRs) [7–10] in connection with the YBE. The basic relations for the components of the representation tensor of a braid group are the spectral parameter independent cases of the YBE (strictly speaking, those having a label permutation).

The aim of this paper is to present in detail an interesting approach to construction of BGRs. In this approach we introduce a weight conservation condition for the S -matrix of the BGR, from which the non-vanishing elements of the S -matrix can be determined. With the help of 'weight lattice' analysis of representations of Lie algebra, the structure of BGRs can be obtained without much difficulty. It is then not very difficult to determine these non-vanishing elements explicitly by solving parameter independent YBEs (in the following simply called YBEs). In particular, in terms of the extended Kauffman diagram technique, solving YBEs becomes quite simple in most cases. Therefore it is feasible to find BGRs satisfying the weight conservation condition by solving YBEs directly. In the next section we briefly review the Witten approach for deriving skein relations of link polynomials [10, 11] and give a reasonable definition of the polynomial of a single loop in terms of weight vectors of the irreducible representation of Lie algebra. In section 3 we discuss properties of Markov moves and their constraints on the S -matrix of BGRs. This leads to the introduction of 'weight conservation' condition, which is

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more general than Akutsu Wadati's charge conservation condition. Considering the property of a third type of 'move', we introduce transposition symmetry for the S -matrix. Section 4 is concerned with applications of the weight conservation condition. Finally, we give some remarks in section 5.

2. Definition of the polynomial for a single loop

In our previous paper we discussed the universality of the Witten approach to link polynomials [12]. Here we only give the main results.

In the Witten approach [11], link polynomials are considered as partition functions of Wilson lines. Skein relation of link polynomials can be easily obtained by calculating the related Casimir invariants. Consider the case associated with an irreducible representation R of a given Lie algebra. If the direct product of R decomposes to r distinct irreducible representations of the same Lie algebra,

$$R \otimes R = \sum_{i=1}^r \oplus E_i \tag{1}$$

we will have

$$B^r \psi - \left(\sum_{i=1}^r \lambda_i \right) B^{r-1} \psi + \left(\sum_{i < j}^r \lambda_i \lambda_j \right) B^{r-2} \psi + \dots + (-1)^{r-1} \left(\prod_{i=1}^r \lambda_i \right) \left(\sum_{i=1}^r \lambda_i^{-1} \right) B \psi + (-1)^r \left(\prod_{i=1}^r \lambda_i \right) \psi = 0 \tag{2}$$

where B is the half monodromy operator [6] and λ_i are eigenvalues of B given by

$$\lambda_i = \pm \exp(i\pi(2\Delta_R - \Delta_{E_i})) \tag{3}$$

where the sign $+$ or $-$ corresponds to whether E_i appears symmetrically or antisymmetrically in $R \otimes R$; Δ_R or Δ_{E_i} is the conformal weight of the primary field transforming as R or E_i . This is given in [13] from the Wess-Zumino chiral model:

$$\Delta_R = \frac{C_R}{C_g + k} \quad \Delta_{E_i} = \frac{C_{E_i}}{C_g + k} \tag{4}$$

where C_R or C_{E_i} is the Casimir invariant of the irreducible representation R or E_i , and C_g is that of the adjoint representation of the same Lie algebra. Taking the natural pair of χ with equation (2) and taking into account the framing factor

$$f = \exp(-2\pi i \Delta_R) \\ \langle \chi B^m \psi \rangle = f^m P_{+(m-1)} \tag{5}$$

we will obtain an r th-order skein relation of expectation values of Wilson lines, i.e. an r th-order skein relation of link polynomials.

For the case of the fundamental representation of A_n , the skein relation is a quadratic skein relation

$$q^{-(n+1)} P_{+1} + (q - q^{-1}) P_0 - q^{n+1} P_{-1} = 0. \tag{6}$$

For the cases of fundamental representations of B_n , C_n and D_n , the skein relations are respectively equations (7), (8) and (9):

$$q^{-4n}P_{+2} - (q^{-2n-1} - q^{-2n+1} + 1)P_{+1} - (q^{2n+1} - q^{2n-1} + 1)P_0 + q^{4n}P_{-1} = 0 \tag{7}$$

$$q^{-4n-2}P_{+2} - (q^{-2n-2} - q^{-2n} - 1)P_{+1} + (q^{2n+2} - q^{2n} - 1)P_0 - q^{4n+2}P_{-1} = 0 \tag{8}$$

$$q^{-4n+2}P_{+2} - (q^{-2n} - q^{-2n+2} + 1)P_{+1} - (q^{2n} - q^{2n-2} + 1)P_0 + q^{4n-2}P_{-1} = 0. \tag{9}$$

Equations (7)–(9) are cubic skein relations. For the fundamental representation of G_2 one can obtain a quartic skein relation; similarly, further higher-order skein relations can be obtained from various non-fundamental irreducible representations of Lie algebra.

The polynomial of a single loop can be determined from a quadratic skein relation. Using the property $P[oo] = P[o]P[o]$, we obtained the polynomial of a single loop for the case of the fundamental representation of A_n from equation (6):

$$P[o] = \frac{q^{n+1} - q^{-n-1}}{q - q^{-1}} = q^n + q^{n-2} + \dots + q^{-n+2} + q^{-n}. \tag{10}$$

In order to associate with the fundamental representation of A_n , we write equation (10) in terms of its weight vectors as follows:

$$P[o] = \sum_{\Lambda_a} q^{4\Lambda_a \cdot \rho} \tag{11}$$

where Λ_a are weight vectors labelling the fundamental representation of A_n and ρ is half the sum of all positive roots of A_n .

However, the polynomial of a single loop cannot be determined from cubic or higher-order skein relations without further information. This requires the definition of the polynomial of a single loop for cases beyond the fundamental representation of A_n . A very natural definition arises from applying formula (11) to other cases.

3. Markov trace and its constraints on the S-matrix

A free group with generators $\{b_i | i = 1, 2, \dots, n-1\}$ is called the braid group \mathcal{B}_n if the generators satisfy the following defining relations:

$$[b_i, b_j] = 0 \quad \text{for } |i - j| > 1 \tag{12}$$

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}. \tag{13}$$

\mathcal{B}_n is also called a braid group because it has a simple realization on n strings defined by

$$b_i \longleftrightarrow \begin{array}{c} 1 \quad i-1 \quad i \quad i+1 \quad n \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ \dots \quad \vdots \quad \vdots \quad \vdots \quad \dots \end{array} \longleftrightarrow \begin{array}{c} 1 \quad i-1 \quad i \quad i+1 \quad n \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ \dots \quad \vdots \quad \vdots \quad \vdots \quad \dots \end{array} \tag{14}$$

Under these correspondences, the defining relations (12) and (13) become the topological equivalence identities of braid. The linear representation of \mathcal{B}_n is the isomorphism

$$g : \mathcal{B}_n \rightarrow \text{End}\left(\prod_{i=1}^n \otimes V\right) \tag{15}$$



given by

$$g_i := g(b_i) = I^{(1)} \otimes I^{(2)} \otimes \dots \otimes I^{(i-1)} \otimes S \otimes I^{(i+2)} \otimes \dots \otimes I^{(n)} \tag{16}$$

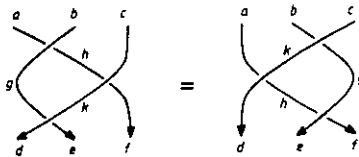
where $I^{(j)}$ is an $N \times N$ unit matrix at the j th position and S is in an $N^2 \times N^2$ matrix. In terms of definition (16), the defining relations (12) hold automatically and equation (13) becomes the following explicit relation for the elements of the S -matrix:

$$S_{gh}^{ab} S_{kf}^{hc} S_{de}^{gk} = S_{kg}^{bc} S_{dh}^{ak} S_{ef}^{hg}. \tag{17}$$

This is just the spectral parameter independent YBE. According to Kauffman's notation

[14]  stands for S_{cd}^{ab} and  for $(S^{-1})_{cd}^{ab}$, and equation (17) is expressed

conveniently as



$$\tag{18}$$

Now we consider the case that V is a carry space of some irreducible representation of any Lie algebra. It is known that the canonical basis of V is labelled by weight vectors of the representation under consideration. Let us denote the collection of all weight vectors by W^Λ , and called a weight lattice. In the following discussion, Λ often stands for the highest weight vector and D^Λ denotes the representation characterized by Λ .

We desire such a BGR that the closed braid can give topological invariants of links. In order to ensure agreement with the polynomial of a single loop defined in the last section (refer to [8] and [7]), we define the Markov trace by

$$\Phi(A) = \text{tr}(Hg(A))$$

$$H = \prod_{i=1}^n \otimes h \tag{19}$$

where $A \in \mathcal{B}_n$, the symbol 'tr' stands for the standard trace of a matrix and

$$h = (h_b^a)$$

$$h_b^a = q^{4\Lambda_a \cdot \rho} \delta_b^a. \tag{20}$$

The property of the Markov move I $\Phi(A_1 A_2) = \Phi(A_2 A_1)$ requires that

$$\text{tr}(Hg(A_1)g(A_2)) = \text{tr}(Hg(A_2)g(A_1)). \tag{21}$$

It is sufficiently guaranteed if

$$[H, g_i] = 0. \tag{22}$$

By using equations (16) and (20), equation (22) is written as

$$(q^{4(\Lambda_a + \Lambda_b) \cdot \rho} - q^{4(\Lambda_c + \Lambda_d) \cdot \rho}) S_{cd}^{ab} = 0. \tag{23}$$

This gives the weight conservation condition

$$\Lambda_a + \Lambda_b = \Lambda_c + \Lambda_d \tag{24}$$

i.e. the elements of $S = (S_{cd}^{ab})$ must vanish unless condition (24) is satisfied. Strictly speaking, equation (24) is only a sufficient condition for the first factor in equation (23) being zero. However, the choice of equation (20) is not unique and, in fact, ρ can be replaced by the sum of ρ and some root vectors. Independent of choice, the condition (24) is necessarily required.

The property of the Markov move II is the following constraints on the diagonal elements of the S -matrix:

$$\sum_b S_{ab}^{ab} q^{A_{b \cdot \rho}} \quad \text{independent of } a \tag{25}$$

if the non-vanishing elements S_{cd}^{ab} also satisfy $a + b = c + d$.

The formula of the link polynomial takes the same form as that in [7]:

$$P(A) = (\tau \cdot \bar{\tau})^{-(n-1)/2} \left(\frac{\bar{\tau}}{\tau} \right)^{e(A)/2} \Phi(A). \tag{26}$$

We define a star operator by

$$*b_i = b_i \tag{27}$$

satisfying the property

$$*(A_1 A_2) = *A_2 *A_1 \quad \forall A_1, A_2 \in \mathcal{B}_n \tag{28}$$

and define a 'prime' operator by

$$b'_i = b_{n-i} \tag{29}$$

satisfying the property

$$(A_1 A_2)' = A'_1 A'_2 \quad \forall A_1, A_2 \in \mathcal{B}_n. \tag{30}$$

We notice that the closed braid of $A, A', *A$ and $*A'$ are isotopic to each other, so they must give the same polynomial. Since $A' = T^{-1}AT$, where T is an element of \mathcal{B}_n indicated by

$$T = \prod_{j=1}^{n-1} \prod_{i=1}^{n-j} b_i \tag{31}$$

which corresponds to



then $\Phi(A') = \Phi(T^{-1}AT) = \Phi(A)$ is satisfied identically. Now $\Phi(A) = \Phi(*A)$ requires that

$$S_{cd}^{ab} = S_{ab}^{cd}. \tag{32}$$

This is easy to check by means of the Kauffman diagram notation. Therefore, the S -matrix is assumed to have transposition symmetry as well as a weight conservation condition.

4. Applications

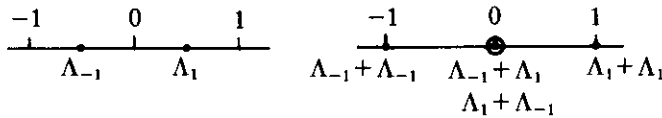
For a given irreducible representation D^Λ of a Lie algebra, we have the corresponding weight lattice $W^\Lambda = \{\Lambda_a | a \in l \subset \mathbb{Z}\}$. The weight lattice of the direct product representation $D^\Lambda \otimes D^\Lambda$ is the superposition of W^Λ , i.e. $\{\Lambda_a + \Lambda_b | \forall \Lambda_a, \Lambda_b \in W^\Lambda\}$.

In general, $D^\Lambda \otimes D^\Lambda$ is not irreducible, and many weight vectors of it may be degenerate. It means that several weight vectors coincide at one point.

From the configuration of the weight lattice of $D^\Lambda \otimes D^\Lambda$ we can determine the structure of the corresponding S -matrix of the BGR under the weight conservation condition: any m -fold degenerated weight vectors give an $m \times m$ non-zero submatrix. These submatrices contribute non-zero elements to the whole S -matrix. In the following, we apply this approach to some concrete cases.

4.1. A_l

We know that the weight lattice of $D^{[1/2]}$ has two weights (left diagram). The weight lattice of $D^{[1/2]} \otimes D^{[1/2]}$ is obtained as the superposition of $W^{[1/2]}$ (right diagram):



where a single dot stands for a single weight vector and one dot surrounded by one (or $m - 1$) circle stands for two (or m) weight vectors coinciding at the same point. From the right diagram, we have the following structure of the S -matrix:

$$(S_{cd}^{ab}) = \begin{matrix} & \begin{matrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{matrix} & \begin{bmatrix} * & & & \\ & * & * & \\ & * & * & \\ & & & * \end{bmatrix} \\ \begin{matrix} 1 & 1 & -1 & -1 \end{matrix} & & \end{matrix} \quad (33)$$

where the marked positions can be non-zero; the remaining positions must be zero due to the weight conservation condition.

Similarly, for the cases of higher-dimensional representations $D^{[j]}$ (j is half of an integer) we have

$$(S_{cd}^{ab}) = \text{block diag}(\sigma_1 \sigma_2 \dots \sigma_{2j+1} \dots \sigma_2 \sigma_1) \quad (34)$$

where $\sigma_m (m = 1, 2, \dots, 2j + 1)$ is an $m \times m$ matrix without vanishing elements.

4.2. Fundamental representation of rank-two Lie algebra

For the rank-two Lie algebra (A_2, B_2, C_2, D_2 and G_2) the weight lattice can be drawn on a plane, so the weight lattice of $D^\Lambda \otimes D^\Lambda$ can be obtained easily, i.e. setting the origin of W^Λ at each weight lattice point of W^Λ and drawing the lattice W^Λ once we obtain the weight lattice of $D^\Lambda \otimes D^\Lambda$. Then the structure of the S -matrix relating to the fundamental representation of A_2, B_2, C_2, D_2 and G_2 can be determined without much difficulty. The results have been given in a previous paper [10] (for more detail, see [15]). For brevity, we omit it here.

4.3. A_n, B_n, C_n and D_n (fundamental representation)

The weight lattice of $D^\Lambda \otimes D^\Lambda$ can be obtained from that of D^Λ pictorially for rank-two (even some rank-three) Lie algebras. However, for higher-rank Lie algebras, it is impossible to do this pictorially. This can be overcome by introducing the following symbolism:

$$\sum_{\Lambda_a \in W^\Lambda} d_a e^{-\Lambda_a} \tag{35}$$

where the integer d_a is the degeneracy number of weight vector Λ_a . With the symbolic form (35), we have a one-to-one correspondence between the symbolic polynomial (35) and weight lattice configuration in weight space. Therefore the superposition of weight lattice can be realized via the standard multiplication rules (distributive rules).

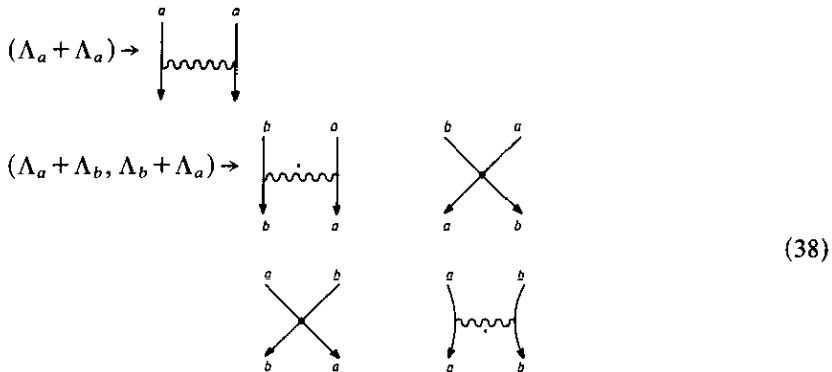
The weight lattice of the fundamental representation of A_n is a collection of $n + 1$ vectors,

$$W^{[1,0,\dots,0]} = \{\Lambda_a | a = n, n-2, \dots, -n+2, -n\} \tag{36}$$

satisfying $\Lambda_a + \Lambda_b \neq \Lambda_{a'} + \Lambda_{b'}$ unless $a = a', b = b'$ for $a < b, a' < b'$. We observe that

$$\left(\sum_a e^{-\Lambda_a}\right)^2 = \sum_a e^{-(\Lambda_a + \Lambda_a)} + \sum_{a < b} 2e^{-(\Lambda_a + \Lambda_b)}. \tag{37}$$

This demonstrates that there are two sorts of weight vectors in the weight lattice of $D^{[1,0,\dots]} \otimes D^{[1,0,\dots]}$: single-fold and two-fold. The non-vanishing contributions to the S -matrix are the following Kauffman 'state' diagrams:



We can now write the S -matrix which satisfies the weight conservation condition in terms of a Kauffman diagram:

$$\text{Crossing} = u_c \left[\text{Wavy} + v_{a,b}^b \text{Crossing} + w_{a,b}^b \text{Wavy} + w'_{a,b} \text{Wavy} \right] \tag{39}$$

where $u_a, p_{a+b}^b, w_{a+b}^b$ and w'_{a+b} are coefficients to be determined by the YBE, and $p_c^b = p_c^{c-b}$ due to transposition symmetry of the S -matrix.

Now we consider the case of the fundamental representation of B_n . The weight lattice is a collection of $2n + 1$ vectors

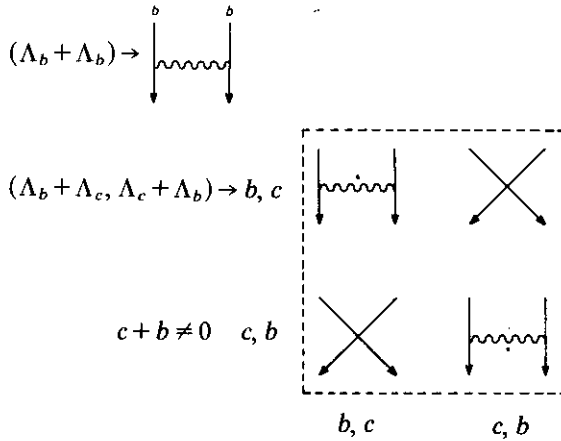
$$W^{[1,0,\dots,0]} = \{\Lambda | b = 2n, 2n-2, \dots, -2n+2, -2n\} \tag{40}$$

satisfying $\Lambda_a + \Lambda_b \neq \Lambda_{a'} + \Lambda_{b'}$ unless $a = a', b = b'$ or $a + b = a' + b' = 0$ for $a < b, a' < b'$.

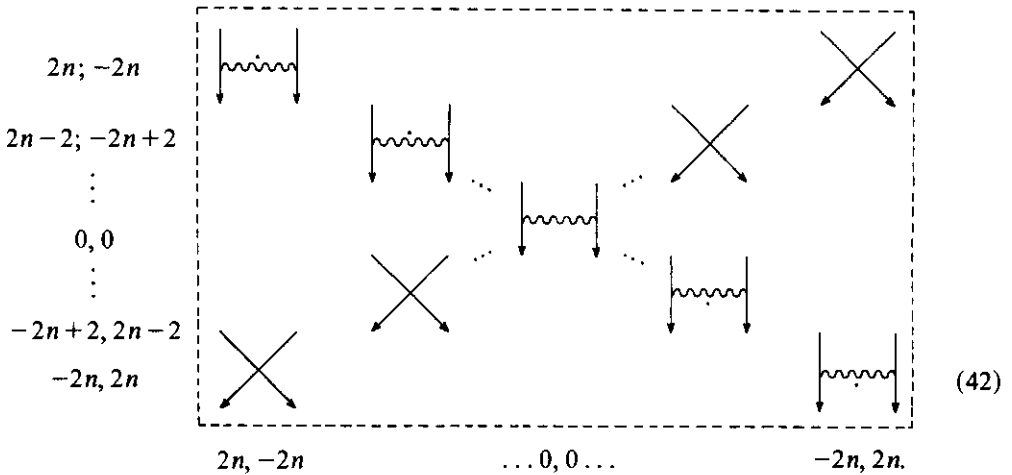
In this case we have

$$\begin{aligned} \left(\sum_b e^{-\Lambda_b}\right)^2 &= \left(\sum_{b \neq 0} e^{-\Lambda_b}\right)^2 + 2 \sum_{b \neq 0} e^{-\Lambda_0 - \Lambda_b} + e^{-2\Lambda_0} \\ &= \sum_{b \neq 0} e^{-(\Lambda_b + \Lambda_b)} + \sum_{\substack{b < c \\ b+c \neq 0}} 2e^{-(\Lambda_b + \Lambda_c)} + (2n+1)e^{-o} \end{aligned} \tag{41}$$

where $\Lambda_a + \Lambda_{-a} = 0$ have been used. The first term in equation (41) shows single-fold weight vectors, the second term shows two-fold weight vectors and the third term shows one $(2n+1)$ -fold weight vectors. The non-vanishing contributions to the S -matrix are easily obtained:



$(\{\Lambda_a + \Lambda_{-a} | a \in I\}) \rightarrow$



In the above $(2n+1) \times (2n+1)$ square array, every non-diagonal and non-skew diagonal position has the contribution of such 'states' as

$$a \pm b \neq 0. \tag{43}$$

Then the S -matrix is written as

(44)

where $q_b^a = 0$ unless $a \pm b \neq 0$; $p_c^b = p_c^{-b}$ and $q_b^a = q_{-a}^{-b}$ due to transposition symmetry of the S -matrix.

The cases of the fundamental representations of C_n and D_n are discussed similarly. The results are the same as equation (44) except that the sets of labels for C_n and D_n are

$$l = \{2n - 1, 2n - 3, \dots, 1, -1, \dots, -2n + 3, -2n + 1\}. \tag{45}$$

4.4. The case that the set of labels has vacancies

We knew in section 2 that the weight conservation condition guaranteed Markov move I and that the label conservation guaranteed Markov move II simply to be equation (25). In previous cases, the sets of labels which label the elements of the S -matrix have equal intervals and, fortunately, the non-vanishing elements of the S -matrix under weight conservation satisfy label conservation. But this is not always true, especially in the cases of higher-level representations. Because the label conservation condition contains less constraints than the weight conservation condition, we can always rechoose labels so that the non-vanishing elements of the S -matrix under weight conservation also satisfy label conservation. In general, the set of labels no longer has equal intervals. It is evident that the labels of weight vectors should be chosen in such a way that the intervals of labels of weight vectors in the same direction of any simple root are the same and those in different directions of various simple roots may be different.

The simplest non-trivial one among symmetric tensor representations of A_n is the six-dimensional representation of A_2 . The set of labels having the least vacancies is

$$l = \{3, 1, 0, -1, -2, -3\} \tag{46}$$

which is chosen in such a way that the intervals of labels of weight vectors along the direction of α_1 is the twice of that along α_2 . The α 's are simple roots. The structure of the S -matrix is

$$S = \text{block diag}(\sigma_1^{(6)} \sigma_2^{(4)} \sigma_2^{(3)} \sigma_3^{(2)} \sigma_4^{(1)} \sigma_5^{(0)} \sigma_4^{(-1)} \sigma_5^{(-2)} \sigma_4^{(-3)} \sigma_3^{(-4)} \sigma_2^{(-5)} \sigma_1^{(-6)}). \tag{47}$$

The details of the submatrices have been given in a previous paper [10].

Among antisymmetric tensor representations of A_n , the simplest non-trivial one is the 10-dimensional representation of A_4 (because that of A_2 is just the conjugate representation of the fundamental representation of A_2 ; that of A_3 is isomorphic to the fundamental representation of D_3). After some simple calculations we find that when the intervals of labels of weight vectors along the directions of $\alpha_1, \alpha_2, \alpha_3$ and α_4 are 3, 1, 2 and 2 respectively, the set of labels has the least vacancies and takes

$$l = \{11, 9, 5, 3, 1, -1, -3, -5, -7, -11\}. \tag{48}$$

If this set is adopted, the S -matrix has a block diagonal structure with 22 submatrices. The non-vanishing elements of these submatrices can be determined by the weight

conservation condition. The details of these submatrices and the final result of the BGR were given in [16] by two of the authors (Ge, Xue) with others.

5. Remarks

The weight conservation condition we propose is a constraint on the S -matrix by Markov move I, but the YBE is in fact the constraint by the defining relations of the Braid group. One may wonder if those two constraints are always compatible. It is worth noting that the compatibility between those two constraints is not accidental. The defining relations of the braid group can be derived from a Markov trace and the star 'prime' operation as a simple sufficient relation. Therefore, it is reliable to take the elements of the S -matrix which do not satisfy the weight conservation condition as zero before solving the YBE.

For the cases of fundamental representations of A_n, B_n, C_n, D_n and G_2 , and of a few non-fundamental ones, the set of labels with equal intervals can guarantee the non-vanishing elements of the S -matrix to also satisfy the label conservation condition. So the structure of the BGR (S -matrix) is the following block diagonal form:

$$S = \text{block diag}(\sigma_1 \sigma_2 \dots \sigma_N \dots \sigma_2 \sigma_1) \quad (49)$$

where the $i \times i$ submatrix $\sigma_i (i = 1, 2, \dots, N)$ has further vanishing elements. While for the cases of most non-fundamental representations, the set of labels no longer has equal intervals. So the block structure is different from equation (49), which depends on a concrete configuration of the set of labels.

It is worthwhile mentioning that the BGR obtained in our approach has transposition symmetry (see equation (23)), and thus it can be diagonalized through a similar transformation by an orthogonal matrix. This is essential to recent Yang-Baxter-ization approaches [17].

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References

- [1] Yang C N 1967 *Phys. Rev. Lett.* **19** 1312; 1968 *Phys. Rev.* **168** 1920
- [2] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)
- [3] Zamolodchikov A B and Zamolodchikov Al B 1978 *Nucl. Phys. B* **133** 525; 1979 *Ann. Phys.* **120** 253
- [4] Faddeev L D 1982 *Recent Advances in Field Theory and Statistical Mechanics Les Houches Session XXX IX* ed J-B Zuber and R Stora (Amsterdam: Elsevier) p 261
- [5] Yang C and Ge M L (ed) 1989 *Braid Group, Knot Theory and Statistical Mechanics* (Singapore: World Scientific)
- Jimbo M (ed) 1990 *Yang-Baxter Equation in Integrable Systems* (Singapore: World Scientific)

- [6] Moore G and Seiberg N 1988 *Phys. Lett.* **212B** 451
 Alvarez-Gaume L, Gomez C and Sierrea G 1989 *Phys. Lett.* **220B** 142
 Frohlich J 1987 *Nonperturbative Quantum Field Theory* ed G 't Hooft, A Jaffe, G Mack, P K Mitter and R Stora (Plenum) p 71
- [7] Akutsu Y and Wadati M 1987 *J. Phys. Soc. Japan* **56** 838; 1988 *J. Phys. Soc. Japan* **56** 3039; 1988 *Commun. Math. Phys.* **117** 243
- [8] Reshetikhin N Yu 1987 *LOMI Preprints* E-4-87, E-17-87
- [9] Ge M L, Wang L Y, Xue K and Wu Y S 1989 *Int. J. Mod. Phys. A* **4** 335
- [10] Ge M L, Li Y Q and Xue K 1990 *J. Phys. A: Math. Gen.* **23** 605
 Ge M L, Li Y Q, Wang L Y and Xue K 1990 *J. Phys. A: Math. Gen.* **23** 619
- [11] Witten E 1989 *Commun. Math. Phys.* **121** 3351
- [12] Ge M L, Li Y Q and Xue K 1990 *Int. J. Mod. Phys. A* **5** 1975
- [13] Kniznik V G and Zamolodchikov A B 1984 *Nucl. Phys. B* **247** 83
- [14] Kauffman L H 1987 *Topology* **26** 395; 1987 State models for knot polynomials—an introduction *Proc. Meeting of the Brazilian Mathematical Society (July 1987)*
- [15] Li Y Q 1989 New approach to invariants of link and representations of Braid group *Ph.D thesis* Lanzhou University
- [16] Ge M L, Gwa L H, Piao F and Xue K 1990 BGR associated with the ten-dimensional representation of SU(5) and its Yang-Baxterization *Preprint* ITP-SB-90-02, Stony Brook
- [17] Ge M L, Wu Y S and Xue K 1990 Explicit trigonometric Yang-Baxterization *Preprint* ITP-SB-90-01, Stony Brook